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Limiting parameter values for switch-on and switch-off shocks in ideal MHD

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Abstract. We investigate under which parameter regimes the MHD Rankine-Hugoniot conditions, which describe discontinuous solutions to the MHD equations, allow for switch-on and switch-off shock solutions. We derive limiting values which agree with the literature and show how we can visualize these limits in the parameter space spanned by Alfvén Mach number and plasma beta. We show that the superposition of a switch-on and a switch-off shock is also an MHD shock, such that the magnetic field is aligned with the shock normal.

Key Words. MHD, shocks, Rankine-Hugoniot

Introduction

The dynamical behaviour of plasmas is often described by the equations of ideal magnetohydrodynamics (MHD). Whereas the stationary Euler equations only have the isotropic sound speed as a characteristic speed, the stationary MHD system has three highly anisotropical characteristic speeds: the slow magnetosonic speed, the Alfvén speed and the fast magnetosonic speed, which makes the MHD system much more rich and complex.

The MHD system is highly nonlinear and allows for large-amplitude waves. In the wave steepening limit, these solutions become discontinuous. The mathematical description of MHD discontinuities is given by the *Rankine-Hugoniot* (RH) jump conditions. The MHD discontinuities can be classified in (i) *linear discontinuities* and (ii) *MHD shocks*. This does not imply that all solutions to the RH jump conditions are physically admissible.

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The Alfvén speed plays a central role in ideal MHD, and MHD shocks can connect sub-Alfvénic flow to super-Alfvénic flow. These solutions to the RH conditions are called *intermediate shocks*.

The existence of these intermediate shocks in still under debate. Let us first summarize the main arguments against the existence of intermediate shocks. Landau & Lifschitz [14] performed a classical stability analysis and showed that intermediate shocks are unstable with respect to small perturbations. Also Falle & Komissarov [8] reject the existence of intermediate shocks. They argue that wave steepening would lead to compound waves instead of intermediate shocks. Most intermediate shocks cross even more than one characteristic speed. On the other hand, Coppi [3] countered some of these objections by noticing that the ideal MHD system is not strictly hyperbolic. Wu [19], De Sterck et al. [7], Delmont & Keppens [6] and many other authors have found intermediate shocks in numerical simulations. Amongst other observers, Chao et al. [2] and Feng & Wang [9] claim to have observed intermediate shocks in respectively Voyager 2 and Voyager 1 data.

An alternative manner to connect a sub-Alfvénic state to a super-Alfvénic state would be by a compound wave. These compound waves can consist of a slow shock which travels with its maximal propagation speed and a rarefaction fan directly attached to it. Brio & Wu [1] detected those compound waves in numerical simulations which have become classical test problems for numerical codes. Another type of compound signal consists of a slow shock layer, immediately followed by a rotational discontinuity (Whang et al. [18]).

Recently, Goedbloed [11] classified the MHD shocks by rewriting the RH equations in the de Hoffmann-Teller frame (de Hoffmann & Teller [12]) introducing the existence of a distinct time reversal duality between entropy-allowed and entropy-forbidden solutions. Delmont & Keppens [5] revisited the classical RH conditions and augment these results in terms of the commonly exploited shock parameters in any shock frame. We found parameter ranges for intermediate shocks. These parameter ranges can be interesting to study the behavior of an intermediate shock, e.g. if one performs a numerical simulation involving intermediate shocks (as in e.g. Delmont & Keppens [6]).

In this paper we summarize the findings of that paper, and focus on the switch-on and switch-off solutions to the RH jump conditions in greater depth. A switch-on shock can be seen as the transition between a fast and an intermediate shock, while a switch-off shock is the transition case between an intermediate and a slow shock. These switch-on and switch-off shocks are proved to be stable with respect to small perturbations (theoretically by Todd [17] and numerically by Chu & Taussig [4]).

Solving the Rankine-Hugoniot conditions

Governing equations

The set-up of our problem is the following. Given is a known state $\mathbf{u}_k$, connected to an unknown state $\mathbf{u}_u$ by a stationary MHD shock. We will show that there exist at most three real possibilities for the value of $\mathbf{u}_u$. As mentioned above, discontinuous solutions to the ideal MHD equations should satisfy the RH jump conditions. Defining the flux term $\mathbf{F} = (\rho v_n, \rho v_n^2 + p + \frac{B_t^2}{2}, \rho v_n v_t - B_n B_t, v_n(\frac{\gamma}{\gamma-1}p + \rho\frac{v_n^2+v_t^2}{2} + B_n^2) - B_n B_t v_t, v_n B_t - v_t B_n, B_n)$, in any frame where the shock is stationary (including the de Hoffmann-Teller frame), the MHD RH conditions become

$$\mathbf{F}_u = \mathbf{F}_k.$$

Index n refers to the direction of the shock normal, and index t refers to the tangential vector components in the plane spanned up by $\mathbf{B}_u$ and $\mathbf{B}_k$. Further, ρ is the mass density, $\mathbf{v}$ is the velocity, p the thermal pressure and $\mathbf{B}$ the magnetic field. The ratio of specific heats, γ, is considered a constant parameter, as we will assume an ideal gas equation of state. For a derivation of these well-known expressions, we refer to De Hoffmann & Teller [12]; Jeffrey & Taniuti [13]; Liberman & Velikhovich [16], Goedbloed & Poedts [10]. Since MHD shocks are essentially $2D$, the system is reduced to a 6×6-system.

We introduce the parameters plasma beta

$$\beta \equiv \frac{2p}{B_n^2 + B_t^2},$$

inclination to the shock normal

$$\theta \equiv \frac{B_t}{B_n},$$

and the Alfvén Mach number

$$M \equiv \sqrt{\frac{\rho v_n^2}{B_n^2}}.$$

It is possible to recover the primitive variables from these parameters (see e.g. Delmont & Keppens [5]). We note $\mathbf{u}_u = (M_u, \theta_u, \beta_u)$, and $\mathbf{u}_k = (M, \theta, \beta)$, where we have dropped the index k.

Now, define $\xi \equiv ((M^2-1)\theta, 2M^2 + \beta(1+\theta^2) + \theta^2, (\frac{\gamma}{\gamma-1}\beta + M^2)(1+\theta^2)M^2)$. We showed in [5] that the stationary MHD RH system can be reduced to a 3×3-system, namely

$$\xi_u = \xi_k.$$

MHD shock types: classical $1-2-3-4$ *classification*

Whereas in HD, a certain state can only be sub- or supersonic, ideal MHD has three highly anisotropical characteristic speeds: the slow magnetosonic speed v_s, the normal Alfvén speed a_n and the fast magnetosonic speed v_f. The full set of MHD equations is hyperbolic, but non-strictly hyperbolic.: $0 \leq v_s \leq a_n \leq v_f$. These three characteristic speeds divide states into four categories. Superfast states ($|v_n| > v_f$) are called 1-states, subfast states ($a_n < |v_n| < v_f$) are 2-states and superslow ($v_s < |v_n| < a_n$) and subslow ($|v_n| < v_s$) states are respectively called 3-states and 4-states. A shock is said to be of *shock type* $i \to j$ if it connect an upstream i-state to a downstream j-state . It is well-known (see e.g. Libermann & Velikhovich [16]) that once the shock type and one state is given, if a solution exists, it must be unique.

We call a shock admissible if it satisfies the second law of thermodynamics: entropy should increase during the passage of a shock, and admissible versus inadmissible shocks can be related through the time duality principle from Goedbloed [11]. When the upstream state is of type i and the downstream state is of type j, then the shock type is $i \to j$. Furthermore, in terms of these shock types, the admissibility condition translates as $i < j$.

Switch-on & Switch-off shocks

Assume now that $\xi_{1,k} = 0$. This means that the known state is exactly Alfvénic in the stationary shock frame ($M = 1$) or that the magnetic field of this state is aligned with the shock normal ($\theta = 0$). Suppose the state $\mathbf{u}_k$ is connected through an MHD shock to an unknown state $\mathbf{u}_u$. Then $\xi_{1,u}$ should also vanish. Therefore, solutions which are chracterized by $\xi_1 = 0$ can be classified as follows:

- $1 \to 4$ shocks with $\theta_u = \theta_k = 0$. Although these shocks connect a sub- to a super-Alfvénic state, they are stable. We will refer to this shock as an HD-shock.
- $2 = 3 \to 2 = 3$ rotational waves. These solutions are linear discontinuities. They satisfy $\theta_u = -\theta_k$, such that the magnetic field flips around the shock normal.
- $1 \to 2 = 3$ shocks are called *switch-on* shocks. The upstream magnetic field is aligned with the shock normal, and the downstream is exactly Alfvénic.
- $2 = 3 \to 4$ shocks are called *switch-off* shocks. The downstream magnetic field is aligned with the shock normal, and the downstream is exactly Alfvénic.

General solution

Given is a state $\mathbf{u}_k$ in the stationary shock frame. Solving the RH conditions leads to the following unknown state $\mathbf{u}_u$:

$$
\begin{aligned}
M_u &= \sqrt{\frac{(M^2-1)\theta+\psi}{\psi}}, \\
\beta_u &= \frac{\left((\gamma-1)((\theta-\psi)^2+(1+\theta^2)\beta)-4M^2\right)(M^2-1)+2M^2(\psi\theta+\psi^2)}{(M^2-1)(\gamma+1)(1+\psi^2)}, \\
\theta_u &= \psi,
\end{aligned}
$$

where ψ satisfies the cubic equation:

$$C(\psi) \equiv \psi^3+\tau_2\psi^2+\tau_1\psi+\tau_0 = 0,$$

with its coefficients given by

$$
\begin{aligned}
\tau_2 &= -\theta\left((\gamma-1)(M^2-1)-M^2\right), \\
\tau_1 &= (M^2-1)\left((\gamma-1)(M^2-1)+\gamma(\beta(\theta^2+1)+\theta^2)-2\right), \\
\tau_0 &= -(\gamma+1)\theta(M^2-1)^2.
\end{aligned}
$$

Note that this only holds when $M \neq 1$. Let us now consider the 2 singular cases: $\theta = 0$ and $M = 1$.

In the specific case, where $\theta = 0$, one HD solution, and potentially also two switch-on or switch-off solutions, exist. Those two solutions are essentially the same, except for the sign of θ_u. The HD shock solution is given by

$$(M_u, \theta_u, \beta_u) = \left(\sqrt{\frac{(\gamma-1)M^2+\gamma\beta_1}{\gamma+1}}, 0, \frac{4M^2-(\gamma-1)\beta}{\gamma+1}\right),$$

while the two switch-on or switch-off solutions are given by

$$
\begin{aligned}
M_u &= 1, \\
\beta_u &= \frac{1-2M^2-\beta}{(\gamma-1)M^4+\gamma(\beta-2)M^2-\gamma(\beta-1)} - 1, \\
\theta_u &= \pm\sqrt{((\gamma\beta-2)-(\gamma-1)(M^2-1))(M^2-1)}.
\end{aligned}
$$

Finally, we consider the irregular case where $M = 1$. In this case a rotational solution ($(M_u, \theta_u, \beta_u) = (M, -\theta, \beta)$) and both a switch-on and a switch-off solution exist, where $\mathbf{u}_u(\mathbf{u}_k)$ is the inverse function of the switch-on and switch-off solutions described above, namely:

$$
\begin{aligned}
M_u &= \sqrt{\frac{\sqrt{A} \pm 2\gamma(\beta+1)(\theta^2+1)}{2(\gamma+1)}},\\
\beta_u &= \beta(1+\theta^2) + \theta^2 - 2(M_u^2 - 1),\\
\theta_u &= 0,
\end{aligned}
$$

where

$$A = 2\gamma^2\beta\theta^2 + \gamma^2\beta^2 + 2\gamma^2\beta^2\theta^2 + \gamma^2\beta^2\theta^4 + 2\gamma^2\beta\theta^4 + \gamma^2\theta^4 - 4\gamma\beta - 4\gamma\beta\theta^2 + 4 + 4\theta^2.$$

As we will show later on, these switch-on and switch-off unknown states, can also be connected through a stationary HD shock.

Results

Restrictions to the existence of switch-on and switch-off shocks

As mentioned earlier, solving the RH conditions reduces to solving a cubic equation. Therefore, there are always one or three real solutions. Lax [15] has shown that when only one real solution exists, it must be a $1 \to 2$ or a $3 \to 4$ solution. It is well-known that the cubic C has three real solutions if and only if

$$\Omega \equiv 27\tau_0^2 + 4\tau_1^3 + 4\tau_2^2\tau_0 - \tau_2^2\tau_1^2 - 18\tau_2\tau_1\tau_0 < 0.$$

On the other hand, when $\Omega > 0$, there is only one non-trivial solution. In the transition case where $\Omega = 0$, there exist two distinct non trivial solutions.

Since both τ_0 and τ_1 have (M^2-1) as a factor, $M^2 - 1 = 0$ implies that $\Omega = 0$. Therefore, there are two distinct solutions with $\theta_u \neq \theta$.

On the other hand, when $\theta = 0$, we have that

$$\Omega = 1728(M^2-1)^3((\gamma-1)M^2 + \gamma(\beta-1) - 1)^3.$$

This implies that switch-on shocks can only exist when

$$1 < M < \sqrt{\frac{\gamma(1-\beta)+1}{\gamma-1}},$$

and switch-off shocks can only exist when

$$\sqrt{\frac{\gamma(1-\beta)+1}{\gamma-1}} < M < 1.$$

This restiction is only valid when $\frac{\gamma(1-\beta)+1}{\gamma-1} > 0$, i.e. when $\beta < \frac{\gamma+1}{\gamma}$. These results agree with the literature (see e.g. De Sterck et al. [7]).

Another non-trivial requirement it that β_u should be positive. For the switch-on and switch-off shocks, this requirement reduces to

$$\frac{1-2M^2-\beta}{(\gamma-1)M^4+\gamma(\beta-2)M^2-\gamma(\beta-1)} > 1,$$

which is satisfied whenever

$$0 < \beta < \frac{(\gamma-1)M^4-2\gamma M^2+(\gamma-1)}{\gamma M^2-\gamma+1}.$$

A mathematical solution to the HD shocks always exists, the only requirement here is the positive β requirement,which now reduces to

$$0 < \beta < \frac{4}{\gamma-1}M^2.$$

For $\gamma = \frac{5}{3}$, all these observations are summarized in Fig. 1.

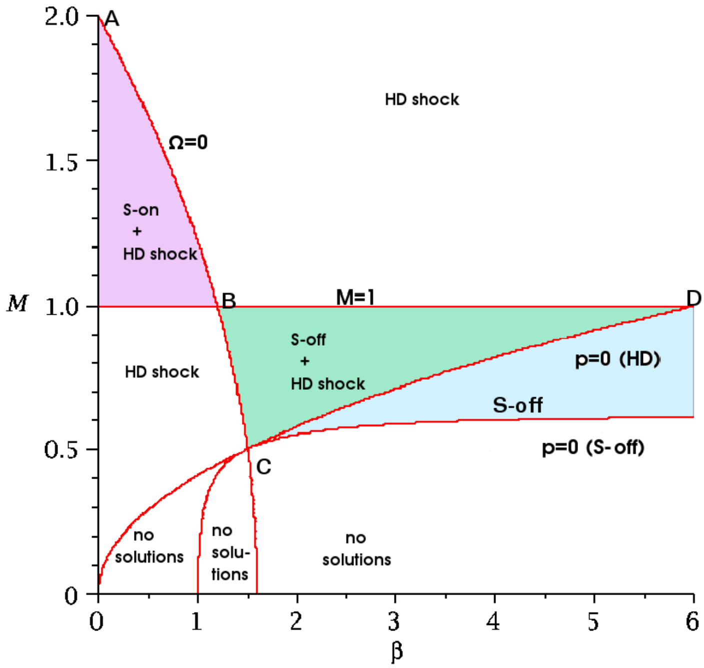

Fig. 1. Shown is the $\theta = 0$ plane of the (M, θ, β) parameter space. The color coding refers to different regions where certain shock types are possible.

Limiting values for switch-on and switch-off shocks

We will now use Fig. 1 to derive limiting values for switch-on and switch-off shocks.

- *Point A.* The maximum Mach number at which switch-on shocks can be found is reached in point A (see Fig. 1). Filling out $\beta = 0$ in $\Omega = 0$, and solving for M, leads us to conclude that the maximum Mach number for switch-on shocks equals $\sqrt{\frac{\gamma+1}{\gamma-1}}$.
- *Point B.* Note that the coordinates of point B are given by $(\beta, M) = \left(\frac{2}{\gamma}, 1\right)$. This can be found by filling out $M = 1$ in $\Omega = 0$. Note that at $\beta = \frac{2}{\gamma}$, also the transition between magnetically and thermally dominated plasmas takes place.

- *Point C.* We can still derive a minimum value of the Alfvénic Mach number for the existence of switch-off shocks. Therefore we need to find the M coordinate of point C in Fig. 1. We solve the system

$$\frac{(\gamma-1)M^4 - 2\gamma M^2 + (\gamma-1)}{\gamma M^2 - \gamma + 1} = \beta.$$
$$(\gamma-1)M^2 + \gamma(\beta-1) - 1 = 0.$$

  The solution to this equation is $(M, \theta) = (\sqrt{\frac{\gamma-1}{\gamma+1}}, \frac{4}{\gamma+1})$. Therefore switch-off shocks can only exist when $M > \sqrt{\frac{\gamma-1}{\gamma+1}}$.
- *Point D.* Filling out $M = 1$ in $\beta_u = 0$, gives the coordinates of point D. The solution is $(M, \theta) = (1, \frac{4}{\gamma-1})$. Therefore the maximum plasma-β at which switch-off shocks can occur is $\frac{4}{\gamma-1}$.

We summarize these findings in Table 1.

Table 1. Limiting values for the plasma-β and the Alfvénic Mach number.

parameter	Switch-on	Switch-off	Switch-on ($\gamma = \frac{5}{3}$)	Switch-off ($\gamma = \frac{5}{3}$)
$M_{\min}$	1	$\sqrt{\frac{\gamma-1}{\gamma+1}}$	1	0.5
$M_{\max}$	$\sqrt{\frac{\gamma+1}{\gamma-1}}$	1	2	1
$\beta_{\min}$	0	$\frac{2}{\gamma}$	0	1.2
$\beta_{\max}$	$\frac{2}{\gamma}$	$\frac{4}{\gamma-1}$	1.2	6

HD shock as superposition of switch-on and switch-off shock

The RH conditions are equivalent to equations $\xi_u = \xi_k$, which express the existence of three shock invariants. Therefore two states can be connected through the stationary RH conditions if and only if they have the same value for the expression $\xi_1 \equiv (M^2 - 1)\theta, \xi_2 \equiv 2M^2 + \beta(1 + \theta^2) + \theta^2$ and $\xi_3 \equiv (\frac{\gamma}{\gamma-1}\beta + M^2)(1 + \theta^2)M^2$. Denoting the relation "state A can be connected to state B through the stationary RH conditions"as $A \mapsto_{RH} B$, this relation $\mapsto_{RH}$ is an equivalence. Indeed:

- $\mapsto_{RH}$ is reflexive: $A \mapsto_{RH} A$. Every state can be connected to itself through the stationary RH conditions.

- $\mapsto_{RH}$ is symmetric: $A \mapsto_{RH} B \Rightarrow B \mapsto_{RH} A$. If state A can be connected to state B through the stationary RH conditions, then also state B can be connected to state A by these conditions. Of course only one of these connections satisfies the entropy condition.
- $\mapsto_{RH}$ is transitive: $A \mapsto_{RH} B \wedge B \mapsto_{RH} C \Rightarrow A \mapsto_{RH} C$. Indeed: if $A \mapsto_{RH} B$, then $\xi_i(A) = \xi_i(B)$, and if $B \mapsto_{RH} C$, then $\xi_i(B) = \xi_i(C)$. Hence $A \mapsto_{RH} B \wedge B \mapsto_{RH} C$ implies $\xi_i(A) = \xi_i(B) = \xi_i(C)$, which means that $A \mapsto_{RH} C$.

Therefore, an HD shock with $\Omega < 0$, can be seen as the superposition of a switch-on and a switch-off shock, since in that case, the equivalence class, containing the up- and downstream states of the HD shocks, also contains two exactly Alfvénic states.

Conclusion

We have derived limiting values of the plasma parameters at which the Rankine-Hugoniot jump conditions allow for switch-on and switch-off shocks. We have shown that the superposition of a switch-on and a switch-off shock leads to an HD shock.

References

[1] Brio, M. & Wu, C. C. 1988 An upwind differencing scheme for the equations of ideal magnetohydrodynamics. J. Comp. Phys. **75**, 400–422.

[2] Chao, J. K., Lyu, L. H., Wu, B.H., Lazarus, A. J., Chang, T.S. & Lepping, R.P. 1993 Observation of an intermediate shock in interplanetary space. J. Geophys. Res. **98**, 17443–17450.

[3] Coppi, P. S., Blandford, R. D. & Kennel, C. F. 1988 On the existance and stability of intermediate shocks ESA SP-285, 381-384

[4] Chu, C. K. & Taussig, R. T. 1967 Numerical Experiments of Magnetohydrodynamic Shocks and the Stability of Switch-On Shocks. Phys. Fluids **10**, 249–256.

[5] Delmont, P. & Keppens, R. 2010 Parameter ranges for slow, intermediate and fast MHD shocks. J. Plasma. Phys. doi:10.1017/S0022377810000115.

[6] Delmont, P. & Keppens, R. 2010 Shock refraction in ideal MHD. Journal of Physics: Conference Series, accepted.

[7] De Sterck, H., Low, B. C. & Poedts, S. 1998 Complex magnetohydrodynamic bow shock topology in field-aligned low-β flow around a perfectly conducting cylinder Phys. of Plasmas **11**, 4015–4027.

[8] Falle, S. A. E. G. & Komissarov, S. S. 1997 On the existence of intermediate shocks. Mon. Not. R. Astron. Soc. **123**, 265–277.

[9] Feng, H. & Wang, J. M. 2008 Observations of a 2 $\longrightarrow$ 3 Type Interplanetary Intermediate Shock. Solar Phys. **247**, 195–201.

[10] Goedbloed, H. & Poedts, S. 2004 Principles of Magnetohydrodynamics With Applications to Laboratory and Astrophysical Plasmas. Cambridge University Press.

[11] Goedbloed, J.P. 2008 Time reversal duality of magnetohydrodnamical shocks. Phys. Plasmas **15**, 062101

[12] De Hoffmann, F. and Teller, E. 1950 Magneto-Hydrodynamic Shocks. Phys. Rev. **80**, 692–703.

[13] Jeffrey, A. & Taniuti, T. 1964 Nonlinear Wave Propagation. Academic Press, New York.

[14] Landau, L. D. & Lifschitz, E. M. 1959 Fluid Mechanics. Permagon press, Oxford.

[15] Lax, P. D. 1957 Hyperbolic System of Conservation Laws II. Comm. Pure Appl. Math. **10**, 537–566.

[16] Liberman, M. A. & Velikhovich, A. L. 1986 Physics of Shock Waves in Gases and Plasmas. Springer

[17] Todd, L. 1965 Evolution of switch-on and switch-off shocks in a gas of finite electrical conductivity. J. Fluid Mech. **24**, 597–608.

[18] Whang, Y. C., Zhou J., Lepping, R. P., Szabo, A., Fairfield, D., Kukobun, S., Ogilvie, K. W., Fitzenreiter R. 1998 Double discontinuity : A compound structure of slow shock and rotational discontinuity J. Geophys. Res. **103** 6513–6520.

[19] Wu, C. C. 1987 On MHD intermediate shocks. Geophys. Res. Lett. **14**, 668-671.

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